GENERAL HÖRMANDER AND MIKHLIN CONDITIONS FOR MULTIPLIERS OF BESOV SPACES

RISHAD SHAHMUROV

ABSTRACT. Here a new condition for the geometry of Banach spaces is introduced and the operator–valued Fourier multiplier theorems in weighted Besov spaces are obtained. Particularly, connections between the geometry of Banach spaces and Hörmander-Mikhlin conditions are established. As an application of main results the regularity properties of degenerate elliptic differential operator equations are investigated.

1. INTRODUCTION, NOTATIONS AND BACKGROUND

In recent years, Fourier multiplier theorems in vector–valued function spaces have found many applications in the theory of differential operator equations, especially in maximal regularity of parabolic and elliptic differential–operator equations and embedding theorems of abstract function spaces. Operator–valued multiplier theorems in Banach–valued function spaces have been discussed extensively in [3,8,12, 15,17 and 19]. Boundary value problems (BVPs) for differential–operator equations (DOEs) in H–valued (Hilbert valued space) function spaces have been studied in [1,2,6,7,9,13,14], and the references therein.

$D(\Omega;E)$ will denote the collection of infinitely differentiable E–valued functions with compact support on Ω. Moreover, we denote a bounded and uniformly continuous function spaces with traditional notation BUC^{θ} where

$$\|f\|_{BUC^{\theta}(\Omega;E)} = \sup_{s\in\Omega}\|f(s)\| + \sup_{\substack{t,s\in\Omega\\ s<t}} \frac{\|f(t)-f(s)\|_E}{|t-s|^{\theta}} \text{ for } 0<\theta<1$$

and

$$\|f\|_{BUC^{m+\theta}(\Omega;E)} = \sup_{s\in\Omega}\sum_{k=0}^{m}\left\|f^{(k)}(s)\right\|_E + \sup_{\substack{t,s\in\Omega\\ s<t}} \frac{\left\|f^{(m)}(t)-f^{(m)}(s)\right\|_E}{|t-s|^{\theta}}.$$

Let $S(R^n;E)$ denote the Schwartz class, i.e., a space of E–valued rapidly decreasing smooth functions on R^n. $S^{\dagger}(R^n;E)$ denotes the space of continuous linear operators $L: S\to E$ equipped with the bounded convergence topology sometimes called E–valued tempered distributions. Let $\alpha=(\alpha_1,\alpha_2,\cdots,\alpha_n)$, where α_i are integers. An E–valued generalized function $D^{\alpha}f$ is called a generalized derivative in the sense of Schwartz distributions, if the equality

$$< D^{\alpha}f,\varphi > = \ (-1)^{|\alpha|} < f, D^{\alpha}\varphi >$$

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holds for all $\varphi \in S$. It is known that

$$F(D_x^\alpha f) = (i\xi_1)^{\alpha_1} \cdots (i\xi_n)^{\alpha_n} \hat{f},\ D_\xi^\alpha (F(f)) = F[(-ix_n)^{\alpha_1} \cdots (-ix_n)^{\alpha_n} f]$$

for all $f \in S^\dagger(R^n; E)$.

Let $\mathbf{C}$ be the set of complex numbers and

$$S_\varphi = \{\lambda;\ \lambda \in \mathbf{C},\ |\arg \lambda| \leq \varphi\} \cup \{0\},\ 0 \leq \varphi < \pi.$$

A linear operator A is said to be a φ–positive in a Banach space E, if $D(A)$ is dense in E, and

$$\left\| (A + \lambda I)^{-1} \right\|_{B(E)} \leq M(1 + |\lambda|)^{-1}$$

with $M > 0,\ \lambda \in S_\varphi,\ \varphi \in [0, \pi)$; here I is the identity operator in $E,\ B(E)$ is the space of all bounded linear operators in E. Sometimes instead of $A + \lambda I$, we will write $A + \lambda$ and denote it by A_λ.

Let E be a Banach space and $\gamma = \gamma(x),\ x = (x_1, x_2, \cdots, x_n) \in \Omega \subset R^n$. $L_{p,\gamma}(\Omega; E)$ denotes the space of all strongly measurable E–valued functions that are defined on the measurable subset $\Omega \subset R^n$ with the norm

$$\begin{aligned}
\|f\|_{L_{p,\gamma}(\Omega;E)} &= \left(\int \|f(x)\|_E^p \gamma(x) dx \right)^{\frac{1}{p}},\ 1 \leq p < \infty, \\
\|f\|_{L_{\infty,\gamma}(\Omega;E)} &= \operatorname{ess\,sup}_{x \in \Omega} [\|f(x)\|_E \gamma(x)].
\end{aligned}$$

For $\gamma(x) \equiv 1$, we denote $L_{p,\gamma}(\Omega; E)$ by $L_p(\Omega; E)$. Note that dual of the space $L_{p,\gamma}(\Omega; E)$ is given by $L_{p',\gamma^{-1}}(\Omega; E')$ where $\frac{1}{p} + \frac{1}{p'} = 1$ and $\gamma^{-1}(x) = \frac{1}{\gamma(x)}$.

We shall use Fourier analytic definition of weighted Besov spaces in this study. Therefore, we need to consider some subsets $\{J_k\}_{k=0}^\infty$ and $\{I_k\}_{k=0}^\infty$ of R^N, where

$$J_0 = \left\{ t \in R^N : |t| \leq 1 \right\},\ J_k = \left\{ t \in R^N : 2^{k-1} \leq |t| \leq 2^k \right\} \text{ for } k \in N$$

and

$$I_0 = \left\{ t \in R^N : |t| \leq 2 \right\},\ I_k = \left\{ t \in R^N : 2^{k-1} \leq |t| \leq 2^{k+1} \right\} \text{ for } k \in N.$$

Next, we define the unity $\{\varphi_k\}_{k \in N_0}$ of functions from $S(R^N, R)$. Let $\psi \in S(R, R)$ be nonnegative function with support in $[2^{-1}, 2]$, which satisfies

$$\sum_{k=-\infty}^{\infty} \psi(2^{-k} s) = 1 \text{ for } s \in R \backslash \{0\}$$

and

$$\varphi_k(t) = \psi(2^{-k}|t|),\ \varphi_0(t) = 1 - \sum_{k=1}^{\infty} \varphi_k(t) \text{ for } t \in R^N.$$

Later, we will need the following useful properties:

$$\begin{aligned}
&\operatorname{supp} \varphi_k \subset \bar{I}_k \text{ for each } k \in N_0, \\
&\varphi_k \equiv 0 \text{ for each } k < 0, \\
&\sum_{k=0}^{\infty} \varphi_k(s) \text{ for each } s \in R^N, \\
&J_m \cap \operatorname{supp} \varphi_k = \emptyset \text{ if } |m - k| > 1, \\
&\varphi_{k-1}(s) + \varphi_k(s) + \varphi_{k+1}(s) = 1 \text{ for each } s \in \operatorname{supp} \varphi_k,\ k \in N_0.
\end{aligned}$$

Let $1 \leq q \leq r \leq \infty$ and $s \in R$. The weighted Besov space is the set of all functions $f \in S'(R^N, X)$ for which

$$\|f\|_{B^s_{q,r,\gamma}(R^N,X)} := \left\|2^{ks}\left\{(\check{\varphi}_k * f)\right\}_{k=0}^{\infty}\right\|_{l_r(L_{q,w_q}(R^N,X))} \equiv \begin{cases} \left[\sum\limits_{k=0}^{\infty} 2^{ksr}\|\check{\varphi}_k * f\|^r_{L_{q,\gamma}(R^N,X)}\right]^{\frac{1}{r}} & \text{if } r \neq \infty \\ \sup\limits_{k\in N_0}\left[2^{ks}\|\check{\varphi}_k * f\|_{L_{q,\gamma}(R^N,X)}\right] & \text{if } r = \infty \end{cases}$$

is finite; here q and s are main and smoothness indexes respectively. It is well known that Besov spaces has significant embedding properties. Thus we close section with stating some of them:

$$W_q^{l+1}(X) \hookrightarrow B^s_{q,r}(X) \hookrightarrow W_q^l(X) \hookrightarrow L_q(X) \text{ where } l < s < l+1,$$

$$B^s_{\infty,1}(X) \hookrightarrow BUC^s(X) \hookrightarrow B^s_{\infty,\infty}(X) \text{ for } s \in \mathbf{Z},$$

and

$$B^{\frac{N}{p}}_{p,1}(R^N, X) \hookrightarrow L_\infty(R^N, X) \text{ for } s \in \mathbf{Z}.$$

For more detailed information see [2] and [3]. Let E and E_0 be Banach spaces so that E_0 is continuously and densely embedded in E. We define Besov-Lions spaces as follows:

$$B^{[l],s}_{p,q}(R; E_0, E) = \left\{u : u \in B^s_{p,q}(R; E_0),\ D^{[l]}u \in B^s_{p,q}(R; E)\right\},$$

$$\|u\|_{B^{[l],s}_{p,q}(R;E_0,E)} = \|u\|_{B^s_{p,q}(R;E_0)} + \left\|D^{[l]}u\right\|_{B^s_{p,q}(R,E)} < \infty.$$

We will use this function spaces in embedding theorems and in the study of degenerate elliptic equations.

2. Fourier multipliers

In this section, we shall extend the work of Girardi and Weis [10] which includes many classical multiplier conditions such as Mikhlin and Hörmander. This section is organized in a similar format as [10]. Some new definitions and lemmas will be introduced. In this section X and Y are Banach spaces over the field C and X^* is the dual space of X. The space $B(X, Y)$ of bounded linear operators from X to Y is endowed with the usual uniform operator topology. N_0 is the set of natural numbers containing zero.

It is well known that Fourier transform $F : S(X) \to S(X)$ is defined by

$$(Ff)(t) \equiv \hat{f}(t) = \int_{R^N} \exp(-its)f(s)ds$$

is an isomorphism whose inverse is given by

$$(F^{-1}f)(t) \equiv \check{f}(t) = (2\pi)^{-N} \int_{R^N} \exp(its)f(s)ds,$$

where $f \in S(X)$ and $t \in R^N$.

All the basic properties of F and F^{-1} that hold in the scalar–valued case also hold in vector–valued case; however, the Housdorff–Young inequality need not hold.

Therefore, we need to define similar class of Banach spaces that was introduced by Peetre [11].

Definition 2.1. Let X be a Banach space and $1 \le p \le 2$. We say X has Fourier γ-type p if

$$\|Ff\|_{L_{p',\gamma^{-1}}(R^N,X)} \ \le \ C\|f\|_{L_{p,\gamma}(R^N,X)} \text{ for each } f \in S(R^N,X),$$

where $\frac{1}{p}+\frac{1}{p'}=1,\ F_{p,N}(X)$ is the smallest $C\in[0,\infty]$ and X has Fourier type p if $\gamma = 1$.

Proposition 2.2. Let X be a Banach space with Fourier γ-type $p \in [1,2]$ and $p \le q \le p'$. Then X^* and $L_{q,\gamma}(R^N,X)$ also have Fourier γ-type p provided both are with the same constant $F_{p,N}(X)$.

Proof. It follows directly from the proof of [10, Proposition 2.3], and Fourier γ-type property of X. ∎

Proposition 2.3. Let X be a Banach space with Fourier type $p \in [1,2]$. If $\gamma^{-1} \in L_\infty(R^N)$ then X has a Fourier γ-type p.

Since $\gamma^{-1} \in L_\infty(R^N)$ then there exist $C>0$ such that $1 \le C\gamma(t)$ a.e. Hence we get

$$\|\hat{f}\|_{L_{p',\gamma^{-1}}(R^N,X)} \ \le \ C\left[\int\limits_{R^N}\|\hat{f}(t)\|_X^{p'}dt\right]^{\frac{1}{p'}} \le C^2\|f\|_{L_{p,\gamma}(R^N,X)}. \ ∎$$

It is also possible to show that if $\gamma^{1-\frac{1}{p}} \in L_2$ or $\gamma^{1-\frac{1}{p}} \in L_1$ and $F^{-1}(\gamma^{1-\frac{1}{p}}) \in L_1$ then Proposition 2.3 is still valid.

Proposition 2.4. Let X be a Banach space, and $1 \le p < q$. If

$$\int\limits_{\Omega}\left[\frac{(\tilde{\gamma}(t))^q}{(\gamma(t))^p}\right]^{\frac{1}{q-p}}dt < \infty \tag{1}$$

then $L_{q,\gamma}(\Omega,X) \hookrightarrow L_{p,\tilde{\gamma}}(\Omega,X)$.

Proof. We recall that embedding theorem for L_p spaces is applicable only in bounded domains; however, in weighted case embedding works even in R^N by imposing some conditions on weight. Suppose $f \in L_{q,\gamma}(\Omega,X)$. Then applying generalized Hölder inequality and (1), we complete the proof:

$$\begin{aligned}\|f\|_{L_{p,\tilde{\gamma}}(\Omega,X)} &= \left[\int\limits_{\Omega}\|f(t)\,(\tilde{\gamma}(t))^{\frac{1}{p}}\,\|_X^p dt\right]^{\frac{1}{p}} \\ &\le \|f\|_{L_{q,\gamma}(\Omega,X)}\left(\int\limits_{\Omega}\left[\frac{(\tilde{\gamma}(t))^q}{(\gamma(t))^p}\right]^{\frac{1}{q-p}}dt\right)^{\frac{q-p}{pq}} \le C\|f\|_{L_{q,\gamma}(\Omega,X)}. \ ∎\end{aligned}$$

The following Fourier embedding theorem plays a key role in the proof of multiplier theorem.

Theorem 2.5. Let X be a Banach space with the Fourier γ-type $p \in [1,2]$. Let $1 \le q < p',\ 1 \le r \le \infty$ and $s \ge \frac{N}{u}$ where $\frac{1}{u} = \left(\frac{1}{q}-\frac{1}{p'}\right)$. Assume for each bounded

domain $\Omega \subset R^N$

$$\int_{\Omega} \left[(\gamma(t))^q \left(\tilde{\gamma}(t) \right)^{p'} \right]^{\frac{1}{p'-q}} dt < \infty. \tag{2}$$

Then there exists a constant C depending only on $F_{p,N}(X)$, so that if $f \in B^s_{p,r,\gamma}(R^N, X)$,

$$\left\| \left\{ \hat{f} \cdot \chi_{J_m} \right\}_{k=0}^{\infty} \right\|_{l_r(L_{q,\tilde{\gamma}}(R^N,X))} \leq C \| f \|_{B^s_{p,r,\gamma}(R^N,X)} .$$

Note that Theorem 2.5 remains valid if Fourier transform is replaced by the inverse Fourier transform.

Proof. Let f be in $B^s_{p,r,\gamma}(R^N, X)$. Then for all $k \in N_0$, since $\check{\varphi}_k * f \in L_{p,\gamma}(R^N, X)$ and X has Fourier γ-type p, $\varphi_k \cdot \hat{f} \in L_{p',\gamma^{-1}}(R^N, X)$. By using Proposition 2.4 and (2), we have

$$\hat{f} \cdot \chi_{J_m} = \left(\sum_{k=m-1}^{m+1} \varphi_k \hat{f} \right) \chi_{J_m} \in L_{q,\tilde{\gamma}}(R^N, X).$$

If there exists a constant C_1 so that

$$\| \hat{f} \cdot \chi_{J_m} \|_{L_{q,\tilde{\gamma}}(R^N,X)} \leq C_1 \sum_{k=m-1}^{m+1} 2^{ks} \left\| \hat{f} \cdot \varphi_k \right\|_{L_{p',\gamma^{-1}}(R^N,X)} \tag{3}$$

for each $m \in N_0$ then

$$\begin{aligned} \| \hat{f} \cdot \chi_{J_m} \|_{L_{q,\tilde{\gamma}}(R^N,X)} &\leq C_1 \sum_{k=m-1}^{m+1} 2^{ks} \| F(\check{\varphi}_k * f \|_{L_{p',\gamma^{-1}}(R^N,X)} \\ &\leq C_1 F_{p,N}(X) \sum_{k=m-1}^{m+1} 2^{ks} \| \check{\varphi}_k * f \|_{L_{p,\gamma}(R^N,X)} \end{aligned}$$

and so

$$\left\| \left\{ \hat{f} \cdot \chi_{J_m} \right\}_{m=0}^{\infty} \right\|_{l_r(L_{q,\tilde{\gamma}}(R^N,X))} \leq C F_{p,N}(X) \| f \|_{B^s_{p,r,\gamma}}.$$

It remains to show that (3) holds for some constant C_1. Taking into consideration (2) and applying generalized Hölder's inequality for each $m \in N_0$, we complete the

proof:

$$
\begin{aligned}
\|\hat{f}\cdot\chi_{J_m}\|_{L_{q,\tilde{\gamma}}(X)} &\le \sum_{k=m-1}^{m+1}\left\|\hat{f}\cdot\varphi_k\cdot\chi_{J_m}\right\|_{L_{q,\tilde{\gamma}}(X)} \\
&\le \sum_{k=m-1}^{m+1}\left\|\hat{f}\varphi_k\left[\frac{1+|\cdot|}{4}\right]^{\frac{N}{u}}\chi_{J_m}\right\|_{L_{p',\gamma^{-1}}(X)} \\
&\times \left\|\left[\frac{1+|\cdot|}{4}\right]^{\frac{-N}{u}}\chi_{J_m}\left(\gamma(\cdot)\right)^{\frac{1}{p'}}\left(\tilde{\gamma}(\cdot)\right)^{\frac{1}{q}}\right\|_{L_u(R)} \\
&\le \sum_{k=m-1}^{m+1}\left\|\left[\frac{1+|\cdot|}{4}\right]^{\frac{N}{u}}\chi_{J_m}\right\|_{L_\infty(R)}\left\|\hat{f}\varphi_k\right\|_{L_{p',\gamma^{-1}}(X)} \\
&\times \left[\int_{J_m}\left[\frac{1+|t|}{4}\right]^{-N}\left[\left(\gamma(t)\right)^q\left(\tilde{\gamma}(t)\right)^{p'}\right]^{\frac{1}{p'-q}}dt\right]^{\frac{1}{u}} \\
&\le C\sum_{k=m-1}^{m+1}\left(2^{m-1}\right)^{\frac{N}{u}}\left\|\hat{f}\varphi_k\right\|_{L_{p',\gamma^{-1}}(X)} \\
&\le C\sum_{k=m-1}^{m+1}2^{ks}\left\|\hat{f}\varphi_k\right\|_{L_{p',\gamma^{-1}}(X)}. \quad \blacksquare
\end{aligned}
$$

Corollary 2.6. Let X be a Banach space with Fourier γ-type $p\in[1,2]$. If (2) holds for $q=r=1$ and $r=q=p'$ then the Fourier transform defines bounded operators

$$F:\ B^{N/p}_{p,1,\gamma}(R^N,X)\ \to\ L_{1,\tilde{\gamma}}(R^N,X) \tag{4}$$

$$F:\ B^{0}_{p,p',\gamma^{-1}}(R^N,X)\ \to\ L_{p',\gamma^{-1}}(R^N,X). \tag{5}$$

For a bounded measurable function $m:R^N\to B(X,Y)$, its corresponding Fourier multiplier operator T_m is defined as follows

$$T_m(f)\ =\ F^{-1}[m(\cdot)(Ff)(\cdot)].$$

In this section, we identify conditions on m, extending those of [10], that

$$\|T_0 f\|_{B^s_{q,r,\tilde{\gamma}}}\ \le\ C\|f\|_{B^s_{q,r,\tilde{\gamma}}}\text{ for each }f\in S(X).$$

Definition 2.7. Let $(E(R^N,Z),E^*(R^N,Z^*))$ be one of the following dual systems, where $1\le q,\ r\le\infty$ and $s\in R$

$$(L_{q,\tilde{\gamma}}(Z),L_{q',\tilde{\gamma}^{-1}}(Z^*))\text{ or }(B^s_{q,r,\tilde{\gamma}}(Z),B^{-s}_{q',r',\tilde{\gamma}^{-1}}(R^N,Z^*)).$$

A bounded measurable function $m:R^N\to B(X,Y)$ is called a Fourier multiplier from $E(X)$ to $E(Y)$ if there is a bounded linear operator

$$T_m:\ E(X)\to E(Y)$$

such that

$$T_m(f)\ =\ F^{-1}[m(\cdot)(Ff)(\cdot)]\text{ for each }f\in S(X), \tag{6}$$

$$T_m\text{ is }\sigma(E(X),E^*(X^*))\text{ to }\sigma(E(Y),E^*(Y^*))\text{ continuous}. \tag{7}$$

The uniquely determined operator T_m is the Fourier multiplier operator induced by m.

Remark 2.8. If $T_m \in B(E(X), E(Y))$ and T_m^* maps $E^*(Y^*)$ into $E^*(X^*)$ then T_m satisfies the continuity condition (7).

Lemma 2.9. Suppose $k \in L_{1,\tilde{\gamma}}(R^N, B(X,Y))$ and the weight function $\tilde{\gamma}$ satisfies the following condition:

$$\sup_{t\in R^N} \frac{\tilde{\gamma}(t)}{\tilde{\gamma}(t-s)} \leq C_1 \tilde{\gamma}(s) \text{ for all } s \in R^N.$$

Assume there exist C_2 so that

$$\|k(\cdot)x\|_{L_{1,\tilde{\gamma}}(Y)} \leq \ C_2\|x\|_X \text{ for all } x \in X \tag{8}$$

and C_3 so that

$$\|k^* y^*\|_{L_{1,\tilde{\gamma}}(X^*)} \ \leq \ C_3\|y^*\|_{Y^*} \text{ for all } y^* \in Y^*. \tag{9}$$

Then for $1 \leq q \leq \infty$ the convolution operator

$$K: \ L_{q,\tilde{\gamma}}(R^N, X) \ \rightarrow \ L_{q,\tilde{\gamma}}(R^N, Y)$$

defined by

$$(Kf)(t) \ = \int\limits_{R^N} k(t-s)f(s)ds \text{ for } t \in R^N$$

satisfies $\|K\|_{L_{q,\tilde{\gamma}}\to L_{q,\tilde{\gamma}}} \leq C_1 C_2^{1-\frac{1}{q}} C_3^{\frac{1}{q}}$.

Proof. We shall prove this lemma in a similar manner as [10, Lemma 4.5], applying vector-valued Stein-Weiss interpolation theorem [16, §1.18.5] instead of [12, Theorem 5.1.2]. By using (8), we have the assertion for $q = 1$. Really

$$\begin{aligned}
\|(Kf)(t)\|_{L_{1,\tilde{\gamma}}(Y)} &\leq \int\limits_{R^N}\int\limits_{R^N} \|k(t-s)f(s)\|_Y \tilde{\gamma}(t) dt\, ds \\
&\leq \int\limits_{R^N} \left[\int\limits_{R^N} \|k(t-s)f(s)\|_Y \tilde{\gamma}(t-s)dt\right] \sup_{t\in R^N} \frac{\tilde{\gamma}(t)}{\tilde{\gamma}(t-s)}\, ds \\
&\leq C_1C_2 \int\limits_{R^N} \|f(s)\|_X \tilde{\gamma}(s)\, ds \leq C_1C_2\|f(s)\|_{L_{1,\tilde{\gamma}}(X)}.
\end{aligned}$$

If $f \in L_{\infty,\tilde{\gamma}}(Y)$, $y^* \in Y^*$ and $t \in R^N$ then $\|K\|_{L_{\infty,\tilde{\gamma}}\to L_{\infty,\tilde{\gamma}}} \leq C_1C_3$:

$$\begin{aligned}
| < y^*, (Kf)(t)\tilde{\gamma}(t) >_Y | &\leq \int\limits_{R^N} | < k(t-s)^* y^* \tilde{\gamma}(t), f(s) >_X |ds \\
&\leq \int\limits_{R^N} \|k(t-s)^* y^*\|_{X^*} \tilde{\gamma}(t-s) \frac{\tilde{\gamma}(t)}{\tilde{\gamma}(t-s)} \|f(s)\|_X ds \\
&\leq C_1C_3\|y^*\|_{Y^*}\|f\|_{L_{\infty,\tilde{\gamma}}(X)}.
\end{aligned}$$

In view of [16, §1.18.5], we conclude that $\|K\|_{L_{q,\tilde{\gamma}}\to L_{q,\tilde{\gamma}}} \leq C_1 C_2^{1-\frac{1}{q}} C_3^{\frac{1}{q}}$ for $1 \leq q \leq \infty$. ∎

Proposition 2.10. Let E be a Banach space, $1 \leq p < \infty$ and γ be a positive measurable function on an open subset Ω of R^n, and essentially bounded on a compact subsets of Ω. Then $D(\Omega; E) \hookrightarrow L_{p,\gamma}(\Omega; E)$.

Proof. For $u \in L_{p,\gamma}(\Omega; E)$ and $n \in \mathbf{N}$ let $u_n : \Omega \to E$ be such that

$$u_n = \begin{cases} u(x) & \text{if } \|u(x)\| \leq n \\ 0 & \text{if } \|u(x)\| > n. \end{cases}$$

By the dominated convergence theorem $\lim\limits_{n\to\infty} \|u-u_n\|_{L_{p,\gamma}(\Omega;E)} = 0$, hence a compactly supported function can be approximated by bounded compactly supported functions belonging to $L_p(\Omega; E)$. From the proof of the denseness theorem (classical case), it follows that if u is a compactly supported function belonging to $L_p(\Omega; E)$ then there exists a compact subset $K \subset \Omega$, with $\operatorname{supp} u \subseteq K$, and a sequence of functions $u_n \in D(\Omega; E)$, with supp $u_n \subseteq K$, such that $\lim_{n\to\infty} \|u-u_n\|_{L_p(\Omega;E)} = 0$; since

$$\|u-u_n\|_{L_{p,\gamma}(\Omega;E)} = \left(\int\limits_K \|u(x) - u_n(x)\|^p \gamma(x) dx \right)^{\frac{1}{p}} \leq \left(\sup_{x\in K} \gamma(x) \right)^{\frac{1}{p}} \|u-u_n\|_{L_p(\Omega;E)}$$

we have $\lim\limits_{n\to\infty} \|u - u_n\|_{L_{p,\gamma}(\Omega;E)} = 0.$ ∎

Condition 1. Let p' be a dual pair of p (Fourier γ-type of a Banach spaces X and Y). Suppose γ is measurable on each open subset $\Omega \subset R^N$, essentially bounded on each compact subset $\Omega \subset R^N$ and for each $t \in R^N$:

$$(i) \sup_{t\in R^N} \frac{\tilde{\gamma}(t)}{\tilde{\gamma}(t-s)} \leq C\tilde{\gamma}(s) \text{ for all } s \in R^N; \tag{10}$$

$$(ii) \int\limits_\Omega \left[(\gamma(t))^{1-\frac{1}{p}} \tilde{\gamma}(t) \right]^p dt < \infty \text{ for each } \Omega \subset R^N, \ \operatorname{vol}(\Omega) < \infty.$$

Example 1. (a.) It is easy to see that exponential functions satisfy Condition 1.

(b.) As a second example we can give e.g. polynomials functions of the form $(1 + |x|)^k$.

In [18] author studied FMT in $L_{p,\gamma}(R^N, l_p)$ for $p \in (1, \infty)$. Particularly, it was shown that choosing weight functions in the following form

$$\begin{aligned} (i)\gamma &= |x|^\alpha, \ -1 < \alpha < p-1, \\ (ii)\gamma &= \prod_{k=1}^N \left(1 + \sum_{j=1}^n |x_j|^{\alpha_{jk}} \right)^{\beta_k}, \ \alpha_{jk} \geq 0, \ N \in \mathbf{N}, \ \beta_k \in R \end{aligned}$$

it is possible to establish boundedness of Fourier multiplier operator.

Theorem 2.13. Let X and Y be Banach spaces with Fourier γ-type $p \in [1, 2]$. Assume Condition 1 holds. Then there is a constant C depending only on $F_{p,N}(X)$ and $F_{p,N}(Y)$ so that if

$$m \in B^{\frac{N}{p}}_{p,1,\gamma}(R^N, B(X, Y))$$

then m is a Fourier multiplier from $L_{q,\tilde{\gamma}}(R^N, X)$ to $L_{q,\tilde{\gamma}}(R^N, Y)$ with

$$\|T_m\|_{L_{q,\tilde{\gamma}}(R^N,X)\to L_{q,\tilde{\gamma}}(R^N,Y)} \leq CM_p(m) \text{ for each } q \in [1, \infty] \tag{11}$$

where

$$M_{p,\gamma}(m) \ = \ \inf\left\{ \|m(a\cdot)\|_{B^{\frac{N}{p}}_{p,1,\gamma}(R^N,B(X,Y))} \ : \ a>0 \right\}.$$

Proof. The key points in this proof are the fact (4) and Lemma 2.9. As in the proof of [10,Theorem 4.3] we assume in addition that $m \in S\left(B\left(X,Y\right)\right)$. Hence, $\check{m} \in S\left(B\left(X,Y\right)\right)$. Since, $F^{-1}\left[m\left(a\cdot\right)x\right]\left(s\right) = a^{-N}\check{m}\left(\frac{s}{a}\right)x$ choosing an appropriate a and using (1) we obtain

$$\begin{aligned} \left\|\check{m}\left(\cdot\right)x\right\|_{L_{1,\tilde{\gamma}}(Y)} &= \left\|\left[m\left(a\cdot\right)x\right]^{\vee}\right\|_{L_{1,\tilde{\gamma}}(Y)} \\ \leq \quad & C_1\left\|m\left(a\cdot\right)\right\|_{B^{\frac{N}{p}}_{p,1,\gamma}} \|x\|_X \leq 2C_1 M_{p,\gamma}(m)\left\|x\right\|_X \end{aligned}$$

where C_1 depends only on $F_{p,N}(Y)$. If $m \in S\left(B\left(X,Y\right)\right)$ then $\left[m(\cdot)^*\right]^{\vee} = \left[\check{m}(\cdot)\right]^* \in S\left(B\left(Y^*,X^*\right)\right)$ and $M_{p,\gamma}(m) = M_{p,\gamma}(m^*)$. Thus, in a similar manner as above, we have

$$\left\|\left[\check{m}\left(\cdot\right)\right]^* y^*\right\|_{L_{1,\tilde{\gamma}}(Y)} \leq 2C_2 M_{p,\gamma}(m)\left\|y^*\right\|_{Y^*}$$

for some constant C_2 depends on $F_{p,N}(X^*)$. Since, we have

$$\left\|\check{m}\left(\cdot\right)x\right\|_{L_{1,\tilde{\gamma}}(Y)} \leq 2C_1 M_{p,\gamma}(m)\left\|x\right\|_X$$

and

$$\left\|\left[\check{m}\left(\cdot\right)\right]^* y^*\right\|_{L_{1,\tilde{\gamma}}(Y)} \leq 2C_2 M_{p,\gamma}(m)\left\|y^*\right\|_{Y^*}$$

by Lemma 2.9 we can conclude

$$\left(T_m f\right)(t)) = \int\limits_{R^N} \check{m}\left(t-s\right) f\left(s\right) ds$$

satisfies

$$\left\|T_m f\right\|_{L_{q,\tilde{\gamma}}(R^N,Y)} \leq C M_{p,\gamma}(m)\left\|f\right\|_{L_{q,\tilde{\gamma}}(R^N,X).}$$

Now, taking into account the fact that $S \hookrightarrow B^{\frac{N}{p}}_{p,1,\gamma}$ and using the same reasoning as in the proof of [10,Theorem 4.3] one can easily prove for the general case $m \in B^{\frac{N}{p}}_{p,1,\gamma}$ and that T_m satisfies (7). ∎

Theorem 2.14. Let X and Y be Banach spaces with Fourier γ-type $p \in [1,2]$. Assume Condition 1 holds. Then there exist a constant C depending only on $F_{p,N}(X)$ and $F_{p,N}(Y)$ so that if $m : R^N \to B(X,Y)$ satisfy

$$\varphi_k \cdot m \in B^{\frac{N}{p}}_{p,1,\gamma}(R^N, B(X,Y)) \text{ and } M_{p,\gamma}(\varphi_k \cdot m) \ \leq \ A \tag{12}$$

then m is Fourier multiplier from $B^s_{q,r,\tilde{\gamma}}(R^N,X)$ to $B^s_{q,r,\tilde{\gamma}}(R^N,Y)$ and $\|T_m\| \leq CA$ for each $s \in R$ and $r \in [1,\infty]$.

Proof. By using Theorem 2.13 we shall prove this theorem in a similar manner as [10, Theorem 4.8] . Really, since $\varphi_k \cdot m \in B^{\frac{N}{p}}_{p,1,\gamma}(R^N, B(X,Y))$, Theorem 2.13 ensures that

$$\left\|T_{m\varphi_k} f\right\|_{L_{q,\tilde{\gamma}}(R^N,Y)} \leq C \ M_p(\varphi_k \cdot m) \ \left\|f\right\|_{L_{q,\tilde{\gamma}}(R^N,X)} \leq \ CA\left\|f\right\|_{L_{q,\tilde{\gamma}}(R^N,X)}.$$

In the introduction we defined function $\psi_k = \varphi_{k-1} + \varphi_k + \varphi_{k+1}$ that is equal to 1 on suppφ_k. Thus,

$$\begin{aligned} & \left\| T_{m\psi_k} f \right\|_{L_{q,\tilde{\gamma}}(Y)} \leq \left\| T_{m\varphi_{k-1}} f \right\|_{L_{q,\tilde{\gamma}}(Y)} \\ + \quad & \left\| T_{m\varphi_k} f \right\|_{L_{q,\tilde{\gamma}}(Y)} + \left\| T_{m\varphi_{k+1}} f \right\|_{L_{q,\tilde{\gamma}}(Y)} \\ \leq \quad & 3CA \left\| f \right\|_{L_{q,\tilde{\gamma}}(R^N,X)} . \end{aligned}$$

Let $T_0 : S(X) \to S'(Y)$ be defined as

$$T_0 f = F^{-1} \left[m(\cdot) \left(Ff \right) (\cdot) \right].$$

From the proof of [10, Theorem 4.3] we know that

$$\check{\varphi}_k * T_0 f = T_{m\psi_k} \left(\check{\varphi}_k * f \right).$$

Hence,

$$\begin{aligned} & \left\| \check{\varphi}_k * T_0 f \right\|_{L_{q,\tilde{\gamma}}(Y)} = \left\| T_{m\psi_k} \left(\check{\varphi}_k * f \right) \right\|_{L_{q,\tilde{\gamma}}(Y)} \\ \leq \quad & 3CA \left\| f \right\|_{L_{q,\tilde{\gamma}}(R^N,X)} \end{aligned}$$

and

$$\sum_{k=0}^{\infty} 2^{ksr} \left\| \check{\varphi}_k * T_0 f \right\|^r_{L_{q,\tilde{\gamma}}(Y)} \leq 3CA \sum_{k=0}^{\infty} 2^{ksr} \left\| f \right\|^r_{L_{q,\tilde{\gamma}}(R^N,X)} .$$

Hence, we obtain

$$\left\| T_0 f \right\|_{B^s_{q,r,\tilde{\gamma}}(R^N,Y)} \leq 3CA \left\| f \right\|_{B^s_{q,r,\tilde{\gamma}}(R^N,X)}$$

for $1 \leq q < \infty$. If $q, r < \infty$ then $\mathring{B}^s_{q,r,\tilde{\gamma}} = B^s_{q,r,\tilde{\gamma}}$. Therefore, it remains to show the cases $q = \infty$ and $r = \infty$ and the weak continuity condition (7). The case $r = \infty$ and the weak continuity condition (7) can be proved in a similar manner as [10, Theorem 4.3]. ∎

In the next section, we apply Theorem 2.14 to degenerate DOEs. However, checking assumptions of the theorem for multiplier functions is not practical. Therefore, we prove a lemma that makes Theorem 2.14 more applicable.

Lemma 2.15. Let $\frac{N}{p} < l \in N$, $u \in [p, \infty]$ and

$$\text{where } \frac{1}{p} = \frac{1}{u} + \frac{1}{\tilde{u}}. \tag{13}$$

Moreover, suppose X and Y are Banach spaces having Fourier γ-type $p \in [1, 2]$ and Condition 1 holds. If $m \in C^l(R^N, B(X,Y))$ satisfies the following

$$\|\gamma^{\frac{1}{p}}(\cdot) D^\alpha m(\cdot)|_{I_0}\|_{L_u(B(X,Y))} \ \leq \ A, \ \|\gamma^{\frac{1}{p}}(\cdot) D^\alpha m(2^{k-1}\cdot)|_{I_1}\|_{L_u(B(X,Y))} \ \leq \ A$$

for each $\alpha \in N_0^N, \ |\alpha| \leq l$, then m satisfies conditions of Theorem 2.14.

Proof. By using the fact that $W_{p,\gamma}^{l}(R^{N},B(X,Y))\subset B_{p,1,\gamma}^{\frac{N}{p}}(R^{N},B(X,Y))$ for $\frac{N}{p}<l$ and applying Holder's inequality we get desired result

$$\begin{aligned} M_{p,\gamma}(\varphi_0\cdot m) &\leq K\|\varphi_0 m\|_{W_{p,\gamma}^{l}}\leq K\sum_{|\alpha|\leq l}\sum_{\beta\leq\alpha}\left\|\begin{pmatrix}\alpha\\ \beta\end{pmatrix}D^{\beta}\varphi_0\left(\gamma^{\frac{1}{p}}D^{\alpha-\beta}m\right)\right\|_{L_p}\\ &\leq K\sum_{|\alpha|\leq l}\sum_{\beta\leq\alpha}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}\left\|D^{\beta}\varphi_0\right\|_{L_{\tilde{u}}(R^{N})}\cdot\sum_{|\alpha|\leq l}\sum_{\beta\leq\alpha}\left\|\gamma^{\frac{1}{p}}D^{\alpha}m|_{I_0}\right\|_{L_u}\\ &\leq KAC_{\varphi_0}\end{aligned}$$

and

$$\begin{aligned} M_{p,\gamma}(\varphi_k\cdot m) &\leq \|\varphi_k(2^{k-1}\cdot)m(2^{k-1}\cdot)\|_{B_{p,1,\gamma}^{\frac{N}{p}}}=\|\varphi_1(\cdot)\,m(2^{k-1}\cdot)\|_{B_{p,1,\gamma}^{\frac{N}{p}}}\\ &\leq K\|\varphi_1(\cdot)\,m(2^{k-1}\cdot)\|_{W_{p,\gamma}^{l}}\leq KAC_{\varphi_1}.\end{aligned}$$

∎

We close this section with two very important corollaries that provide different sufficient conditions for $B_{q,r,\tilde{\gamma}}^{s}$-regularity of (6). As a matter of fact these conditions are slightly modified versions of Hörmander and Mikhlin conditions.

Corollary 2.16. (FMT via *Hörmander* condition) Suppose X and Y have Fourier γ-type $p\in[1,2]$ and Condition 1 holds. If $m\in C^{l}(R^{N},B(X,Y))$ satisfies

$$\left[\int\limits_{|t|\leq 2}\|D^{\alpha}m(t)\|^{p}\gamma(t)dt\right]^{\frac{1}{p}}\leq A$$

and

$$\left[R^{-N}\int\limits_{R\leq|t|\leq 4R}\|D^{\alpha}m(t)\|^{p}\gamma(t)dt\right]^{\frac{1}{p}}\leq AR^{-|\alpha|}$$

for each multi–index α with $|\alpha|\leq\left\lceil\frac{N}{p}\right\rceil+1$ then m is Fourier multiplier from $B_{q,r,\tilde{\gamma}}^{s}(R^{N},X)$ to $B_{q,r,\tilde{\gamma}}^{s}(R^{N},Y)$ for each $s\in R$ and $r\in[1,\infty]$.

Proof. Choosing $u=p$ in the Lemma 2.15 we get assertions of corollary. ∎

Corollary 2.17. (FMT via Mikhlin condition) Assume X and Y are Banach spaces with Fourier γ-type $p\in[1,2]$ and Condition 1 holds. If $m\in C^{l}(R^{N},B(X,Y))$ satisfies

$$\left\|\gamma^{\frac{1}{p}}(t)(1+|t|)^{|\alpha|}D^{\alpha}m(t)\right\|_{L_\infty(R^{N},B(X,Y))}\leq A$$

for each multi–index α with $|\alpha|\leq l=\left\lceil\frac{N}{p}\right\rceil+1$, then m is Fourier multiplier from $B_{q,r,\tilde{\gamma}}^{s}(R^{N},X)$ to $B_{q,r,\tilde{\gamma}}^{s}(R^{N},Y)$ for each $s\in R,\ r,\ q\in[1,\infty]$.

Proof. Choosing $u=\infty$ in the Lemma 2.15 one can prove this result in a similar way as [10, Corollary 4.11]. ∎

The following result is a special case of Corollary 2.17. Choosing $\gamma=1$ we obtain a sufficient condition for the multipliers of weighted Besov spaces.

Corollary 2.18. Assume X and Y are Banach spaces with Fourier type p and

$$(i)\ \sup_{t\in R^N}\frac{\tilde{\gamma}(t)}{\tilde{\gamma}(t-s)}\leq C\tilde{\gamma}(s) \text{ for all } s\in R^N$$

$$(ii)\int\limits_{\Omega}[\tilde{\gamma}(t)]^p\,dt<\infty \text{ for each } \Omega\subset R^N,\ \operatorname{vol}(\Omega)<\infty.$$

If $m\in C^l(R^N,B(X,Y))$ satisfies

$$\left\|(1+|t|)^{|\alpha|}D^\alpha m(t)\ \right\|_{L_\infty(R^N,B(X,Y))}\leq\ A$$

for each multi–index α with $|\alpha|\leq l=\left\lceil\frac{N}{p}\right\rceil+1$, then m is Fourier multiplier from $B^s_{q,r,\tilde{\gamma}}(R^N,X)$ to $B^s_{q,r,\tilde{\gamma}}(R^N,Y)$ for each $s\in R,\ r,\ q\in[1,\infty]$.

3. Differential Embeddings

In the present section, by using Corollary 2.18 we shall prove continuity of the following embedding

$$D^\alpha:B^{l,s}_{q,r,\gamma}\left(R^N;E\left(A\right),E\right)\subset B^s_{q,r,\gamma}\left(R^N;E\right).$$

In the next section we will apply above result to non degenerate elliptic equations.

Condition 2. Assume a positive weight function γ satisfies the following:

$$(i)\ \sup_{t\in R^N}\frac{\gamma(t)}{\gamma(t-s)}\leq C\gamma(s) \text{ for all } s\in R$$

$$(ii)\int\limits_{\Omega}[\gamma(t)]^p\,dt<\infty \text{ for each } \Omega\subset R,\ \operatorname{vol}(\Omega)<\infty.$$

Theorem 3.1. Suppose Condition 2 holds and $0<h\leq h_0<\infty$. Let E be a Banach space with Fourier type p and A be a φ-positive operator in E, where $\varphi\in(0\,,\pi]$. If $\alpha=(\alpha_1,\alpha_2,...,\alpha_N)$, $x=\frac{|\alpha|}{l}\leq 1$ and $0<\mu\leq 1-x$ then the following embedding

$$D^\alpha:B^{l,s}_{q,r,\gamma}\left(R^N;E\left(A\right),E\right)\subset B^s_{q,r,\gamma}\left(R^N;E\left(A^{1-x-\mu}\right)\right)$$

is continuous and there exists a positive constant C such that

$$\begin{aligned}&\|D^\alpha u\|_{B^s_{q,r,\gamma}(R^N;E(A^{1-x-\mu}))}\\ \leq\ & C_\mu\left[h^\mu\|u\|_{B^{l,s}_{q,r,\gamma}(R^N;E(A),E)}+h^{-(1-\mu)}\|u\|_{B^s_{q,r,\gamma}(R^N;E)}\right]\end{aligned}$$

for all $u\in B^{l,s}_{q,r,\gamma}\left(R^N;E\left(A\right),E\right)$.

Proof. Since A is constant and closed operator, we have

$$\begin{aligned}\|D^\alpha u\|_{B^s_{q,r,\gamma}(R^N;E(A^{1-x-\mu}))}&=\left\|A^{1-x-\mu}D^\alpha u\right\|_{B^s_{q,r,\gamma}(R^N;E)}\\ &\backsim\left\|F^{-1}\left(i\xi\right)^\alpha A^{1-x-\mu}Fu\right\|_{B^s_{q,r,\gamma}(R^N;E)}.\end{aligned}$$

(The symbol $\backsim$ indicates norm equivalency). In a similar manner, from definition of $B^{l,s}_{q,r,\gamma}\left(R^N;E_0,E\right)$ we have

$$\|u\|_{B^{l,s}_{q,r,\gamma}(R^N;E_0,E)}\sim\|Au\|_{B^s_{q,r,\gamma}(R^N;E)}+\sum_{k=1}^N\left\|F^{-1}\xi_k^l\hat{u}\right\|_{B^s_{q,r,\gamma}}.$$

By virtue of above relations, it is sufficient to prove

$$\begin{aligned}&\left\| F^{-1}\left[(i\xi)^{\alpha} A^{1-x-\mu}\hat{u}\right]\right\|_{B^s_{q,r,\gamma}(R^N;E)}\\ \leq\ & C\left[\left\|F^{-1}A\hat{u}\right\|_{B^s_{q,r,\gamma}(R^N;E)}+\sum_{k=1}^{N}\left\|F^{-1}\left(\xi_k^l\hat{u}\right)\right\|_{B^s_{q,r,\gamma}(R^N;E)}\right].\end{aligned}$$

Hence, the inequality (17) will be followed if we can prove the following estimate

$$\left\| F^{-1}\left[(i\xi)^{\alpha} A^{1-x-\mu}\hat{u}\right]\right\|_{B^s_{q,r,\gamma}(R^N;E)}\leq C\left\|F^{-1}\left(\left[A+I\theta\right]\hat{u}\right)\right\|_{B^s_{q,r,\gamma}(R^N;E)} \tag{14}$$

for all $u\in B^{l,s}_{q,r,\gamma}\left(R^N;E\left(A\right),E\right)$, where

$$\theta=\theta\left(\xi\right)=\sum_{k=1}^{N}\left|\xi_k\right|^l\in S\left(\varphi\right).$$

Let us express the left hand side of (18) as follows

$$\begin{aligned}&\left\| F^{-1}\left[(i\xi)^{\alpha} A^{1-x-\mu}\hat{u}\right]\right\|_{B^s_{q,r,\gamma}(R^N;E)}\\ =\ &\left\|F^{-1}\left(i\xi\right)^{\alpha}A^{1-x-\mu}\left[\left(A+I\theta\right]^{-1}\left[\left(A+I\theta\right)\right]\hat{u}\right\|_{B^s_{q,r,\gamma}(R^N;E)}\end{aligned}$$

(Since A is the positive operator in E and $\theta\left(\xi\right)\in S\left(\varphi\right)$, $\left[\left(A+I\theta\right]^{-1}\right.$ exists). From Corollary 2.18 we know that

$$\begin{aligned}&\left\|F^{-1}\left(i\xi\right)^{\alpha}A^{1-x-\mu}\left[\left(A+I\theta\right]^{-1}\left[\left(A+I\theta\right)\right]\hat{u}\right\|_{B^s_{q,r,\gamma}(R^N;E)}\quad\leq\\ &C\left\|F^{-1}\left(\left[A+I\theta\right]\hat{u}\right)\right\|_{B^s_{q,r,\gamma}(R^N;E)}\end{aligned}$$

holds if operator-function $\Psi\left(\xi\right)=\left(i\xi\right)^{\alpha}A^{1-x-\mu}(A+\theta)^{-1}$ satisfies Mikhlin's condition for each multi–index β, $|\beta|\leq\left[\frac{N}{p}\right]+1$. It is clear that

$$\left\|(1+|\xi|)^{|\beta|}D^{\beta}\Psi\left(\xi\right)\right\|_{L_\infty(B(E))}\quad\leq\quad\sum_{k=0}^{|\beta|}\left\||\xi|^kD^{\beta}\Psi\left(\xi\right)\right\|_{L_\infty(B(E))}$$

Therefore, it is enough to show

$$\left\||\xi|^kD^{\beta}\Psi\left(\xi\right)\right\|_{L_\infty(B(E))}\quad\leq\quad C$$

for $k=0,1,\cdots|\beta|$ and $|\beta|\leq\left[\frac{N}{p}\right]+1$. It is proven in [14] that Ψ satisfies Miklin's condition. Hence proof is completed.■

4. Degenerate differential–operator equations

In this section we study degenerate elliptic DOE

$$(L+\lambda)u\ =\ -\left(\gamma(t)\frac{d}{dt}\right)^2u+A_1(t)\left(\gamma(t)\frac{d}{dt}\right)u+A_\lambda u\ =\ f \tag{15}$$

in $B^s_{q,r}(R;E)$, where $A_\lambda=A+\lambda I$ and $A_1(x)$ are possible unbounded operators in a Banach space E. Let E and E_0 be Banach spaces such that E_0 is continuously

and densely embedded in E. Then

$$B_{p,q}^{[l],s}(R;E_0,E) = \left\{u : u \in B_{p,q}^{s}(R;E_0),\ D^{[l]}u \in B_{p,q}^{s}(R;E)\right\},$$
$$\|u\|_{B_{p,q}^{[l],s}(R;E_0,E)} = \|u\|_{B_{p,q}^{s}(R;E_0)} + \left\|D^{[l]}u\right\|_{B_{p,q}^{s}(R,E)} < \infty$$

denotes the Besov-Lions spaces where

$$D^{[i]} = \left(\gamma(t)\frac{d}{dt}\right)^{i}.$$

Remark 4.1. It is clear that under a substitution

$$\tau = \int_0^t \gamma^{-1}(y)dy \tag{16}$$

spaces $B_{q,r}^{s}(R;E)$ and $B_{q,r}^{[2],s}(R;E(A),E)$, map isomorphically onto the weighted spaces $B_{q,r,\tilde{\gamma}}^{s}(R;E)$ and $B_{q,r,\tilde{\gamma}}^{2,s}(R;E(A),E)$ respectively, where $\tilde{\gamma} = \tilde{\gamma}(\tau) = \gamma(t(\tau))$. Note that, (16) transforms degenerate problem (15) in $B_{q,r}^{s}(R;E)$ to the following non–degenerate problem

$$(L+\lambda)u = -u'' + A_1(t)u' + A_\lambda u = f \tag{17}$$

in $B_{q,r,\tilde{\gamma}}^{s}(R;E)$.

Theorem 4.2. Assume $\tilde{\gamma}$ satisfies the Condition 2. Let E be a Banach space with Fourier type p, A be a φ–positive operator in E for $\varphi \in [0,\pi)$ and

$$A_1(\cdot)A^{-(\frac{1}{2}-\mu)} \in L_\infty(R,B(E)),\ 0 < \mu < \frac{1}{2}.$$

Then for all $f \in B_{q,r,\tilde{\gamma}}^{s}(R;E),\ r,\ q \in [1,\infty]$ and sufficiently large $\lambda \in S(\varphi)$ (17) has a unique solution $u \in B_{q,r,\tilde{\gamma}}^{2,s}(R;E(A),E)$ satisfying coercive estimate

$$\|u''\|_{B_{q,r,\tilde{\gamma}}^{s}(R;E)} + \|A_1u\|_{B_{q,r,\tilde{\gamma}}^{s}(R;E)} + \|Au\|_{B_{q,r,\tilde{\gamma}}^{s}(R;E)} \le C\|f\|_{B_{q,r,\tilde{\gamma}}^{s}(R;E)}. \tag{18}$$

Proof. We first show maximal regularity result for the principal part of (17) i.e.

$$(L_0+\lambda)u = -u'' + A_\lambda u = f.$$

By applying the Fourier transform, we obtain

$$(A+\xi^2+\lambda)u\hat{}(\xi) = f\hat{}(\xi).$$

Since $\xi^2+\lambda \in S(\varphi)$ for all $\xi \in R$ and A is a positive operator, solutions are of the form

$$u(x) = F^{-1}[A+\xi^2+\lambda]^{-1}f\hat{}. \tag{19}$$

By using (19), we get

$$\|Au\|_{B_{q,r,\tilde{\gamma}}^{s}(R;E)} = \left\|F^{-1}A(A+\xi^2+\lambda)^{-1}f\hat{}\right\|_{B_{q,r,\tilde{\gamma}}^{s}(R;E)}$$
$$\|u''\|_{B_{q,r,\tilde{\gamma}}^{s}(R;E)} = \left\|F^{-1}[\xi^2(A+\xi^2+\lambda)^{-1}f\hat{}]\right\|_{B_{q,r,\tilde{\gamma}}^{s}(R;E)}.$$

Therefore, it suffices to show that operator–function

$$\sigma(\xi) = \xi^2(A+\xi^2+\lambda)^{-1},$$

is uniformly bounded multiplier in $B^s_{q,r,\tilde{\gamma}}(R;E)$. Since $\sigma \in C^2(R,B(E))$ and

$$(1+|\xi|)^k D^k \sigma(\xi) \in L_\infty(R,B(E))$$

for each $k=0,1,2$, Corollary 2.18 guarantees us that σ is a uniformly bounded Fourier multiplier in $B^s_{q,r,\tilde{\gamma}}(R;E)$. Thus we obtain

$$\|L_0 u\|_{B^s_{q,r,\tilde{\gamma}}(R;E)} \leq \|u\|_{B^{2,s}_{q,r,\tilde{\gamma}}(R;E(A),E)} \leq C\|(L_0+\lambda)u\|_{B^s_{q,r,\tilde{\gamma}}(R;E)}. \tag{20}$$

The above estimate implies that $L_0+\lambda$ has a bounded inverse acting from $B^s_{q,r,\tilde{\gamma}}(R;E)$ into $B^{2,s}_{q,r,\tilde{\gamma}}(R;E(A),E)$. Next we try to estimate lower order term $L_1 u(t) = A_1(t)u'(t)$. In fact, the Theorem 3.1 ensures that for all $u \in B^{2,s}_{q,r,\tilde{\gamma}}(R;E(A),E)$,

$$\begin{aligned}
\|L_1 u\|_{B^s_{q,r,\tilde{\gamma}}} &\leq \left\|A_1(t)A^{-(\frac{1}{2}-\mu)}\right\|_{L_\infty} \left\|A^{\frac{1}{2}-\mu}u'(t)\right\|_{B^s_{q,r,\tilde{\gamma}}} \\
&\leq C_0 C_\mu \left[h^\mu \|u\|_{B^{2,s}_{q,r,\tilde{\gamma}}(R;E(A),E)} + h^{-(1-\mu)}\|u\|_{B^s_{q,r,\tilde{\gamma}}(R;E)}\right].
\end{aligned}$$

It is also clear that

$$\begin{aligned}
\|u\|_{B^s_{q,r,\tilde{\gamma}}(R;E)} &= \frac{1}{\lambda}\|(L_0+\lambda)u - L_0 u\|_{B^s_{q,r,\tilde{\gamma}}} \\
&\leq \frac{1}{\lambda}\left[\|(L_0+\lambda)u\|_{B^s_{q,r,\tilde{\gamma}}} + \|u\|_{B^{2,s}_{q,r,\tilde{\gamma}}(R;E(A),E)}\right],
\end{aligned}$$

which in its turn implies the following estimate

$$\begin{aligned}
\|L_1 u\|_{B^s_{q,r,\tilde{\gamma}}} &\leq C_1\left[(h^\mu + h^{-(1-\mu)}\frac{1}{\lambda})\|u\|_{B^{2,s}_{q,r,\tilde{\gamma}}(R;E(A),E)}\right. \\
&+ \left. h^{-(1-\mu)}\frac{1}{\lambda}\|(L_0+\lambda)u\|_{B^s_{q,r,\tilde{\gamma}}}\right] \\
&\leq C_1\left[(h^\mu + h^{-(1-\mu)}\frac{1}{\lambda})C\|(L_0+\lambda)u\|_{B^s_{q,r,\tilde{\gamma}}(R;E)}\right. \qquad (21) \\
&+ \left. h^{-(1-\mu)}\tfrac{1}{\lambda}\|(L_0+\lambda)u\|_{B^s_{q,r,\tilde{\gamma}}}\right] \leq \|(L_0+\lambda)u\|_{B^s_{q,r,\tilde{\gamma}}(R;E)} \\
&\times \left[CC_1h^\mu + C_1(C+1)h^{-(1-\mu)}\frac{1}{\lambda}\right].
\end{aligned}$$

Therefore choosing h and λ so that

$$CC_1h^\mu < 1 \text{ and } C_1(C+1)|\lambda|^{-1}h^{-(1-\mu)} < 1$$

we obtain

$$\left\|L_1(L_0+\lambda)^{-1}\right\|_{B\left(B^s_{q,r,\tilde{\gamma}}(R;E)\right)} < 1. \tag{22}$$

In view of (20), (22) and the perturbation theory of linear operators, we conclude that $L+\lambda = (L_0+\lambda)+L_1$ is invertible and its inverse is continuous i.e.

$$(L+\lambda)^{-1} = (L_0+\lambda)^{-1}\left[I + L_1(L_0+\lambda)^{-1}\right]^{-1} : B^s_{q,r,\tilde{\gamma}}(R;E) \to B^{2,s}_{q,r,\tilde{\gamma}}(R;E(A),E).$$

Moreover, combining the estimates (20) and (21) we get (18). Hence proof is completed. ∎

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Department of Mathematics, Yeditepe University, Kayishdagi Caddesi, 34755 Kayishdagi, Istanbul, Turkey

E-mail address: shahmurov@hotmail.com